Numerical Integration of Partial Differential Equations (PDEs)

- Introduction to PDEs.
- Semi-analytic methods to solve PDEs.
- Introduction to Finite Differences.
- Stationary Problems, Elliptic PDEs.
- Time dependent Problems.
- Complex Problems in Solar System Research.

Introduction to PDEs.

- Definition of Partial Differential Equations.
- Second Order PDEs.
  -Elliptic
  -Parabolic
  -Hyperbolic
- Linear, nonlinear and quasi-linear PDEs.
- What is a well posed problem?
- Boundary value Problems (stationary).
- Initial value problems (time dependent).

Differential Equations

- A differential equation is an equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders.
- Ordinary Differential Equation:
  Function has 1 independent variable.
- Partial Differential Equation:
  At least 2 independent variables.

Physical systems are often described by coupled Partial Differential Equations (PDEs)

- Maxwell equations
- Navier-Stokes and Euler equations
  in fluid dynamics.
- MHD-equations in plasma physics
- Einstein-equations for general relativity
- ...
- ...

PDEs definitions

- General (implicit) form for one function u(x,y) :

$$F\left(x, y, u(x,y), \frac{\partial u(x,y)}{\partial x}, \frac{\partial u(x,y)}{\partial y}, \ldots, \frac{\partial^2 u(x,y)}{\partial x \partial y}, \ldots\right) = 0,$$

- Highest derivative defines order of PDE
- Explicit PDE => We can resolve the equation to the highest derivative of u.
- Linear PDE => PDE is linear in u(x,y) and for all derivatives of u(x,y)
- Semi-linear PDEs are nonlinear PDEs, which are linear in the highest order derivative.

Linear PDEs of 2. Order

$$a(x,y)\frac{\partial^2 u(x,y)}{\partial x^2}+b(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y}+c(x,y)\frac{\partial^2 u(x,y)}{\partial y^2}$$

$$+d(x,y)\frac{\partial u(x,y)}{\partial x}+e(x,y)\frac{\partial u(x,y)}{\partial y}+f(u,x,y)=0$$

- $a(x,y)c(x,y) - b(x,y)2 / 4 > 0$ Elliptic
- $a(x,y)c(x,y) - b(x,y)2 / 4 = 0$ Parabolic
- $a(x,y)c(x,y) - b(x,y)2 / 4 < 0$ Hyperbolic

Quasi-linear: coefficients depend on u and/or first derivative of u, but NOT on second derivatives.

PDEs and Quadratic Equations

- Quadratic equations in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

describe cone sections.

- $a(x,y)c(x,y) - b(x,y)2 / 4 > 0$ Ellipse
- $a(x,y)c(x,y) - b(x,y)2 / 4 = 0$ Parabola
- $a(x,y)c(x,y) - b(x,y)2 / 4 < 0$ Hyperbola

With coordinate transformations these equations can be written in the standard forms:

Ellipse: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parabola: $$y^2 = 4ax$$

Hyperbola: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Coordinate transformations can be also applied to get rid of the mixed derivatives in PDEs. (For space dependent coefficients this is only possible locally, not globally)

Linear PDEs of 2. Order

$$a(x,y)\frac{\partial^2 u(x,y)}{\partial x^2}+b(x,y)\frac{\partial^2 u(x,y)}{\partial x \partial y}+c(x,y)\frac{\partial^2 u(x,y)}{\partial y^2}+d(x,y)\frac{\partial u(x,y)}{\partial x}+e(x,y)\frac{\partial u(x,y)}{\partial y}+f(u,x,y)=0$$

- Please note: We still speak of linear PDEs, even if the coefficients a(x,y) ... e(x,y) might be nonlinear in x and y.
- Linearity is required only in the unknown function u and all derivatives of u.
- Further simplification are:
  -constant coefficients a-e,
  -vanishing mixed derivatives (b=0)
  -no lower order derivates (d=e=0)
  -a vanishing function f=0.

Second Order PDEs with more then 2 independent variables

$$Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower order terms} = 0.$$

Classification by eigenvalues of the coefficient matrix:

- **Elliptic:** All eigenvalues have the same sign. [Laplace-Eq.]
- **Parabolic:** One eigenvalue is zero. [Diffusion-Eq.]
- **Hyperbolic:** One eigenvalue has opposite sign. [Wave-Eq.]
- Ultrahyperbolic: More than one positive and negative eigenvalue.

Mixed types are possible for non-constant coefficients, appear frequently in science and are often difficult to solve.

Elliptic Equations

- Occurs mainly for stationary problems.
- Solved as boundary value problem.
- Solution is smooth if boundary conditions allow.

Example: Poisson and Laplace-Equation (f=0)

$$\nabla^2 \Phi = f$$

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \Phi(x) = f(x)$$

Parabolic Equations

- The vanishing eigenvalue often related to time derivative.
- Describes non-stationary processes.
- Solved as Initial- and Boundary-value problem.
- Discontinuities / sharp gradients smooth out during temporal evolution.

Example: Diffusion-Equation, Heat-conduction

$$\frac{\partial}{\partial t}u(x,t) = a \cdot \frac{\partial^2}{\partial x^2}u(x,t) \quad \frac{\partial}{\partial t}u(\vec{r},t) = a \cdot \Delta u(\vec{r},t)$$

Hyperbolic Equations

- The opposite sign eigenvalue is often related to the time derivative.
- Initial- and Boundary value problem.
- Discontinuities / sharp gradients in initial state remain during temporal evolution.
- A typical example is the Wave equation.

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u = 0$$

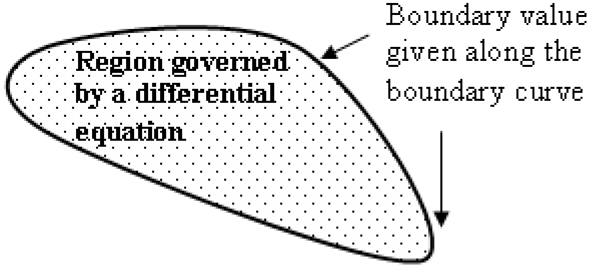
- With nonlinear terms involved sharp gradients can form during the evolution => Shocks

How to solve PDEs?

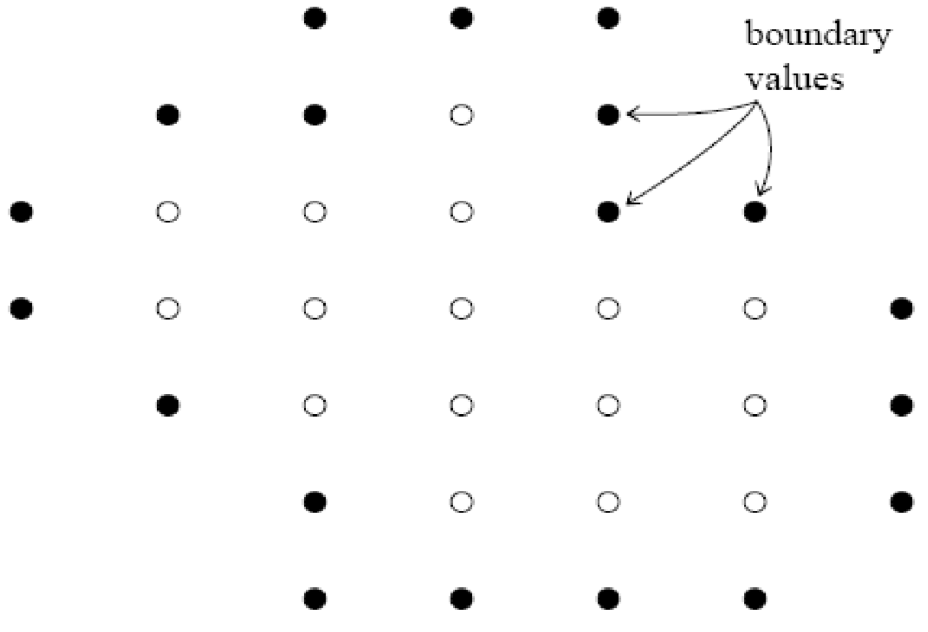
- PDEs are solved together with appropriate **Boundary Conditions** and/or **Initial Conditions**.
- **Boundary value problem**
  -Dirichlet B.C.: Specify u(x,y,...) on boundaries (say at x=0, x=Lx, y=0, y=Ly in a rectangular box)
  -von Neumann B.C.: Specify normal gradient of u(x,y,...) on boundaries.

In principle boundary can be arbitrary shaped.

(but difficult to implement in computer codes)



Boundary value problem

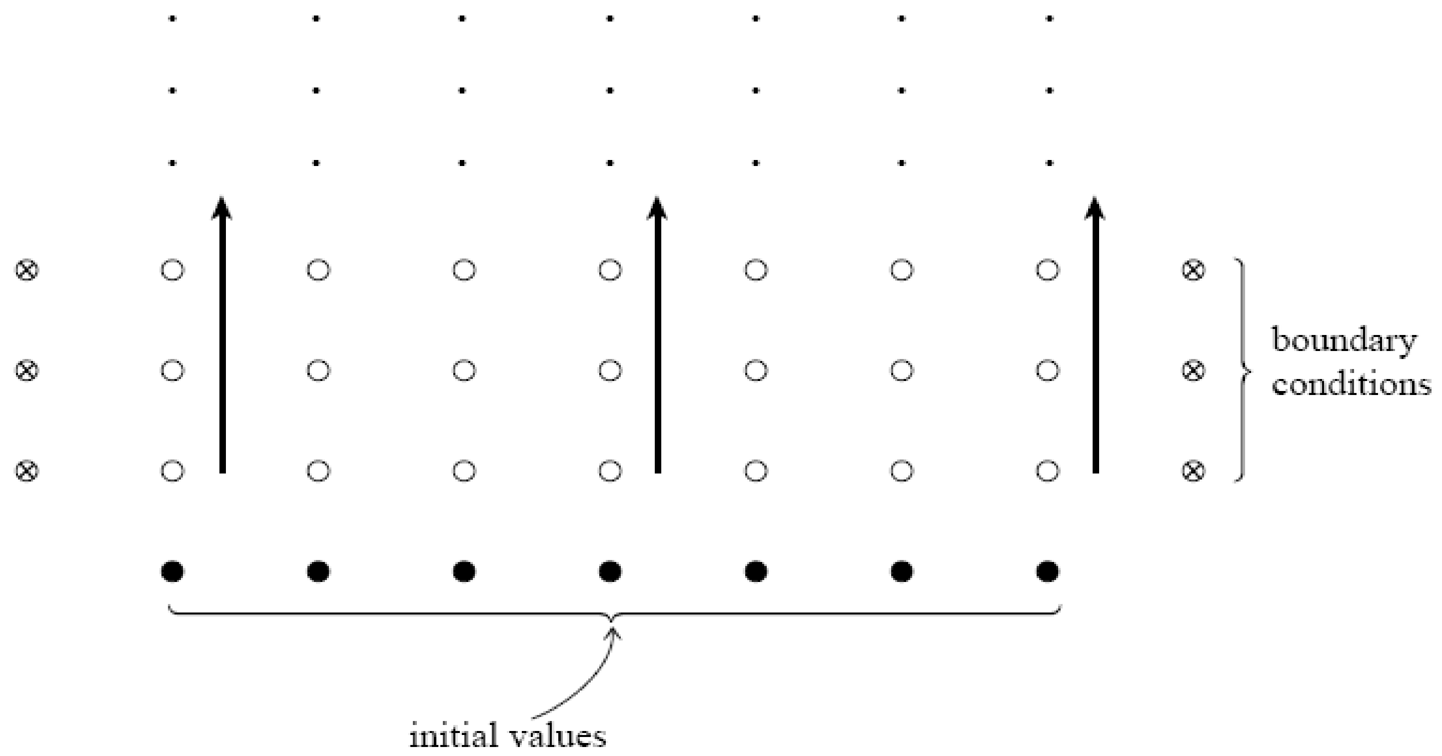


- **Initial value problem**
- Boundary values are usually specified on all boundaries of the computational domain.
- Initial conditions are specified in the entire computational (spatial) domain, but only for the initial time t=0.
- Initial conditions as a Cauchy problem:

  -Specify initial distribution u(x,y,...,t=0)
  [for parabolic problems like the Heat equation]

  - Specify u and du/dt for t=0
  [for hyperbolic problems like wave equation.]

Initial value problem



Semi-analytic methods to solve PDEs.

- From systems of coupled first order PDEs (which are difficult to solve) to uncoupled PDEs of second order.
- Example: From Maxwell equations to wave equation.
- (Semi) analytic methods to solve the wave equation by separation of variables.
- Exercise: Solve Diffusion equation by separation of variables.

How to obtain uncoupled 2. order PDEs from physical laws?

- Example: From Maxwell equations to wave equations.
- Maxwell equations are a coupled system of first order vector PDEs.
- Can we reformulate this equations to a more simple form?
- Here we use the electromagnetic potentials, vectorpotential and scalar potential.